LINEAR ALGEBRA [M. MATH I YEAR]

Back Paper

TOTAL MARKS : 50

Date: December 23, 2025

This exam is of total 50 marks and has a duration of 3 hours (10 AM – 1 PM). Please read all questions carefully. You may use any theorems learned in class, provided you clearly state them before applying.

1. True or false? If a diagonalizable operator has only the characteristic values 0 and 1, it is a projection. Justify your answer. [2 marks]

2. Prove that every operator on an odd-dimensional real vector space has an eigenvalue. [5 marks]

3. Let V be an inner product space with an inner product $\langle .,. \rangle$ and $\|v\| = \sqrt{\langle v, v\rangle}$, for $v \in V$. Then for any vector $\alpha, \beta \in V$ prove that

$$|\langle \alpha, \beta \rangle| \leq \|\alpha\| \|\beta\|.$$

Find when the equality occurs. [5 marks]

4. Let V be $F^{n \times n}$, the space of all $n \times n$ matrices over F(= $\mathbb{C}$ or $\mathbb{R}$). Prove that, for any $A, B \in V$

$$\langle A, B \rangle = \mathrm{trace}(AB^*),$$

defines an inner product on V. Then show that

$$|\mathrm{trace}(AB^*)| \leq \big(\mathrm{trace}(AA^*)\big)^{1/2} \big(\mathrm{trace}(BB^*)\big)^{1/2}.$$

[5 marks]

5. Let V be the space of continuous complex-valued functions on the interval $0 \leq x \leq 1$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)\, \overline{g(x)}\, dx.$$

(a) Find an orthonormal set $\mathcal{S}$ in V. [3 marks]

(b) Prove for any $f \in V$,

$$\sum_{k=-n}^{n} \left| \int_0^1 f(t) e^{-2\pi i k t}\, dt \right|^2 \leq \int_0^1 |f(t)|^2\, dt.$$

[2 marks]

6. Let V be a finite-dimensional inner product space and T a linear operator on V. Show that the range of T^* is the orthogonal complement of the null space of T. [5 marks]

7. Let V be an inner product space and β, γ fixed vectors in V.

(a) Show that $T\alpha = \langle \alpha | \beta \rangle \gamma$ defines a linear operator on V. Show that T has an adjoint, and describe T^* explicitly. [4 marks]

(b) Now suppose V is $\mathbb{C}^n$ with the standard inner product, $\beta = (y_1, \ldots, y_n)$, and $\gamma = (x_1, \ldots, x_n)$. What is the j, k entry of the matrix of T in the standard ordered basis? What is the rank of this matrix? [4 marks]

8. Let V be an inner product space and T a self-adjoint linear operator on V. Then prove that each characteristic value of T is real, and characteristic vectors of T associated with distinct characteristic values are orthogonal. [5 marks]

9. Let f be a symmetric bilinear form on V over the field F, and q be the quadratic form associated with f that is $q : V \to F$ defined by

$$q(\alpha) = f(\alpha, \alpha).$$

Then for $\alpha, \beta \in V$, prove the the polarization identity:

$$f(\alpha, \beta) = \tfrac{1}{4} q(\alpha + \beta) - \tfrac{1}{4} q(\alpha - \beta).$$

[2 marks]

10. Let D be a 2-linear function on the set of all 2×2 matrices A over K with the property that $D(A) = 0$ for all 2×2 matrices A having equal rows, where K is a commutative ring with identity. Then prove that D is alternating. [2 marks]

11. Let K be a commutative ring with identity. If V is a free K-module of rank n, then prove that $M^r(V)$, the set of all r-linear form on V is a free K-module of rank n^r; and show that if $\{f_1, \ldots, f_n\}$ is a basis for the dual module V^*, the n^r tensor products

$$f_{j_1} \otimes \cdots \otimes f_{j_r}, \qquad 1 \leq j_1 \leq n, \ldots, 1 \leq j_r \leq n$$

form a basis for $M^r(V)$. [6 marks]